Heat, Mass, and Momentum Transfer Analogies for the Fully Developed Turbulent Flow of Power Law Fluids in Circular Tubes

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A new correlation is presented to describe heat and mass transfer at large Prandtl or Schmidt numbers to power law fluids in fully developed turbulent flow in a pipe. The resulting expression for the Stanton number differs from earlier semiempirical correlations in that it is based on a continuous eddy viscosity distribution from the wall to the center of the pipe and contains no adjustable parameters to be determined from heat transfer data. Nusselt numbers determined by the new correlation are in excellent agreement with data on heat transfer to both pseudoplastic and dilatant fluids.

Excellent reviews of the available correlations for heat and mass transfer to inelastic, time-independent, non-Newtonian fluids in fully developed turbulent pipe flow are given by Metzner (*18*) and Skelland (*31*). The most widely used correlation to date has been that of Metzner and Friend (*19*), who extended the theoretical analogy of Reichardt (*27*) to purely viscous non-Newtonian fluids to obtain the following equation for the Nusselt number:

$$N_{Nu} = \frac{f/2}{1.20 + c\sqrt{f/2}} N_{Re} N_{Pr} \qquad (1)$$

The parameters of the constitutive equation for the non-Newtonian fluid enter Equation (1) through the friction factor f, the non-Newtonian generalized Reynolds number N_{Re}, the generalized Prandtl number N_{Pr}, and the empirical function c, which in the general case also may be a function of the generalized non-Newtonian Reynolds number. These investigators determined c empirically from experimental heat transfer coefficients for both pseudoplastic and Newtonian fluids. Their final correlation has the form

$$N_{Nu} = \frac{f/2}{1.20 + 11.8\sqrt{f/2}(N_{Prw} - 1)(N_{Prw})^{-1/3}} N_{Re} N_{Pr} \qquad (2)$$

The Prandtl number at the wall N_{Prw} is related to the generalized Prandtl number as follows:

$$\frac{N_{Pr}}{N_{Prw}} = \left(\frac{16}{N_{Ref}}\right)^{\frac{n-1}{n}} \left(\frac{3n+1}{4n}\right) \qquad (3)$$

In developing the above, the rheological behavior of the purely viscous non-Newtonian fluids was represented by the empirical Ostwald-de Waele or power law constitutive equation

$$\tau = -\left\{ K \left| \sqrt{\frac{1}{2}(\Delta : \Delta)} \right|^{n-1} \right\} \Delta \qquad (4)$$

This was chosen in view of its proved ability to portray experimental data over moderately large ranges of shear rate for many systems.

Equation (2) is restricted to nearly isothermal flow. Petersen and Christiansen (*25*) have extended Metzner and Friend's correlation to nonisothermal and transitional flow. These same authors have argued that the Prandtl number to be used in Equation (2) should be corrected for the fact that the effective viscosity in the wall region is less than that in the turbulent core for pseudoplastic fluids. They suggest that the following effective Prandtl number be used in Equation (2):

$$N_{Prc} = \frac{808(n+2)^{(n+2)/(n+1)}}{1{,}055(3n+1)} N_{Prw} \qquad (5)$$

Clapp (*4*) extended Martinelli's (*17*) analogy to power law fluids. However, the Nusselt numbers predicted were considerably lower than those he observed for pseudoplastic fluids in the range $0.7 < n < 0.8$. He proposed the empirical equation

$$N_{Nu} = 0.023\ (9350)^{0.8[1-1/(n)^n]} (N_{Re})^{0.8/(n)^n} (N_{Pr})^{0.4} \qquad (6)$$

Mizushina and Kuriwaki (*20*) recently correlated their heat transfer data on both pseudoplastic and dilatant fluids with a modified form of the Chilton-Colburn (*3*) equation:

$$N_{Nu} = (0.023)^{5(n+2)/[3(4n+1)]} (N_{Re})^{4(n+2)/[3(4n+1)]} (N_{Pr})^{1/3} \qquad (7)$$

Recently, Hughmark (*10*) has extended Longwell's (*15*) equation to power law fluids. The mean velocity profile is

divided into six regions and the empirical universal velocity profiles developed for Newtonian fluids are assumed to apply to power law fluids when recast in terms of the dimensionless variables suggested by Dodge and Metzner (5). The numerical integration of Longwell's equation agrees with the heat transfer data of Metzner and Friend (19) about as well as does Equation (1).

The predictions of these empirical equations for values of the parameters outside the range for which they were developed are doubtful. The theoretical approach of Metzner and Friend (19) appears to be the most promising approach to predicting the Nusselt numbers for purely viscous fluids for the complete range of parameters. In order to put this approach on a more rigorous basis, information on the mean velocity and Reynolds stress distributions near the wall must be obtained. To date no such measurements have been made in the viscous sublayer region adjacent to the tube wall for non-Newtonian fluids. However, recently Wasan, Tien, and Wilke (33) have developed analytical expressions for the velocity and Reynolds stress distributions for the fully developed turbulent flow of Newtonian fluids in tubes. This paper extends the work of these authors to power law fluids. The resulting velocity and Reynolds stress distributions are used in developing the heat and mass transfer analogies for high Prandtl number fluids in a manner analogous to that used by Metzner and Friend (19). However, the resulting expressions for the Nusselt and Stanton numbers involve no adjustable parameters to be determined from heat transfer data. The new correlation compares favorably with data on heat transfer to both dilatant and pseudoplastic fluids.

THEORETICAL ANALYSIS

Consider the fully developed turbulent pipe flow of a power law fluid having constant physical properties. Heat transfer is assumed to occur by both eddy and molecular transport mechanisms. Hence the shear stress τ_{rx} at a plane parallel to the wall and the heat flux q across this plane can be written as the sum of turbulent and molecular fluxes:

$$\frac{\tau_{rx}}{\rho} = -\left[\epsilon_v + \frac{K}{\rho}\Lambda\right]\frac{dU}{dr} \tag{8}$$

where

$$\Lambda = \left|\sqrt{\frac{1}{2}\Delta:\Delta}\right|^{n-1}\left\{1 + \frac{\dfrac{\partial u}{\partial r} + \dfrac{\partial v}{\partial x}}{\dfrac{dU}{dr}}\right\} \tag{9}$$

and

$$q = -\rho C_p(\alpha + \epsilon_c)\frac{dT}{dr} \tag{10}$$

The bar in the definition of Λ denotes a time-averaged quantity. In order to develop the heat transfer analogy, some simplification in this term must be effected. By virtue of the fact that only high Prandtl number fluids are being considered ($N_{Pr} > 1$), and thus the region of interest is very close to the wall, Krantz and Wasan (12) have shown that Λ simplifies to the following form:

$$\Lambda = -[1 + (n-1)A + (n-1)(n-3)B]\left|\frac{dU}{dr}\right|^{n-2}\left(\frac{dU}{dr}\right) \tag{11}$$

where A and B are functions of the Reynolds number which involve time-averaged products of the spatial derivatives of the fluctuating components of the velocity. In the absence of any such data for non-Newtonian fluids, the measurements of Laufer (14) for Newtonian fluids have been employed to estimate the values of these functions. For a Reynolds number of 50,000 the values $A = 0.128$ and $B = 0.0279$ have been obtained. Based upon these estimates Λ can be simplified to the following form:

$$\Lambda \simeq \left|\frac{dU}{dr}\right|^{n-2}\left(\frac{dU}{dr}\right) \tag{12}$$

The estimates used in evaluating the functions in Equation (11) were the maximum values of the time-averaged products in the viscous sublayer. These values would decrease at smaller Reynolds number; thus the values given for A and B are overestimated for most values of the Reynolds number encountered in practice. Therefore Equation (12) should be a very good approximation for all values of the flow behavior index n, provided that the estimated values for A and B are reasonable for power law as well as Newtonian fluids.

It is convenient to redefine the radial coordinate $r = a - y$, and to introduce the dimensionless variables first suggested by Dodge and Metzner (5):

$$u_i^+ = \frac{u_i}{U_*};\ x_i = \frac{x_i(U_*)^{(2-n)/n}}{(K/\rho)^{1/n}} \qquad \begin{matrix}(u_i = u, v, w \text{ or } U)\\ (x_i = x, y \text{ or } \theta)\end{matrix} \tag{13}$$

Upon transformation of coordinates, Equations (8) and (10) become

$$\frac{\tau_{yx}}{\rho} = \left[\frac{\epsilon_v}{\nu} + \phi\Lambda^+\right]\frac{U_*^2}{\phi}\frac{dU^+}{dy^+} \tag{14}$$

where

$$\phi \equiv \left(\frac{fN_{Re}}{16}\right)^{(n-1)/n} \Big/ \left(\frac{3n+1}{4n}\right) \tag{15}$$

Also

$$q = \rho C_p\left[\frac{\epsilon_c}{\rho} + \frac{1}{N_{Pr}}\right]\frac{U_*}{\phi}\frac{dT}{dy^+} \tag{16}$$

The friction factor f is related to the non-Newtonian generalized Reynolds number N_{Re} via the correlation of Dodge and Metzner (5).

Integrating Equation (16) from the wall to some point in the turbulent core where $T = T_a$, and subtracting from it the result of integrating Equation (14) from the wall to some point in the turbulent core where $U = U_a$, one gets, upon rearrangement

$$N_{St} = \frac{\beta f/2}{1 + \sqrt{f/2}\displaystyle\int_0^{y_c^+}\left\{\frac{1}{\left[\dfrac{\overline{uv}^+}{\left(\dfrac{dU^+}{dy^+}\right)} + \dfrac{1}{\beta\phi N_{Pr}}\right]} - \frac{1}{\left[\dfrac{\overline{uv}^+}{\left(\dfrac{dU^+}{dy^+}\right)} + \left(\dfrac{dU^+}{dy^+}\right)^{n-1}\right]}\right\}dy^+} \tag{17}$$

where the Stanton number for heat transfer is defined by

$$N_{St} \equiv \frac{q_w}{\rho C_p (T_a - T_w) U_a} \tag{18}$$

Equation (17) can also be used to predict mass transfer coefficients in the turbulent flow of purely viscous non-Newtonian fluids if the Prandtl number is replaced by the Schmidt number and the Stanton number for mass transfer is defined by

$$N_{Stm} \equiv \frac{N_{Aw}}{(C_a - C_w) U_a} \tag{19}$$

In arriving at Equation (17) several assumptions have been made. First, the shear stress and heat flux at the wall are assumed to be constant. Second, in the region near the wall, $0 \leqq y \leqq y_c$, the shear stress τ_{yx} and the heat transfer flux q are assumed to be equal to their value at the wall. Third, in the fully turbulent core the ratio τ_{yx}/q is assumed to be constant. This assumes that the molecular transport coefficients ν and α can be neglected in the turbulent core, and thus limits the theory to high Prandtl number fluids. Last, the eddy diffusivities of momentum ϵ_v and heat ϵ_c are assumed to be related by $\epsilon_c = \beta\epsilon_v$. The latter is related to the dimensionless Reynolds stress and mean velocity gradient by

$$\frac{\epsilon_v}{\nu} = \frac{\phi \overline{uv}^+}{\left(\frac{dU^+}{dy^+}\right)} \tag{20}$$

The Stanton number N_{St} determined by Equation (17) will be independent of axial distance along the tube, provided the first assumption above holds true. Under these conditions the temperature profiles will stabilize several diameters downstream such that $T_a - T_w$ will be constant. Siegel and Sparrow (29) have shown that this condition also will be approximately true for the case of constant wall temperature rather than constant heat flux at the wall.

For $n = 1$, Equation (17) reduces to the expression for the Stanton number obtained by Wasan and Wilke (34) for Newtonian fluids. Note also that Equation (17) is of the same form as the correlation of Metzner and Friend given by Equation (1).

In order to evaluate the definite integral in Equation (17) it is necessary to know the mean velocity and Reynolds stress distributions in the vicinity of the pipe wall. In the following section, theoretical expressions for these distributions will be obtained for power law fluids.

The approach followed here will be similar to that used by Wasan, Tien, and Wilke (33) for Newtonian fluids. The time-averaged equations of motion are used to relate the Reynolds stress $\overline{uv}$ to the mean velocity U. An analytical expression is obtained for the latter by requiring that a Taylor series expansion for U in the wall region give a smooth and continuous transition to the universal logarithmic velocity distribution in the turbulent core. The method is considerably complicated in the present case because of the nonlinearity introduced by the constitutive equation.

The time-smoothed equations of mean motion for the fully developed turbulent incompressible flow of a power law fluid can be written in cylindrical coordinates as follows (1):

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \tag{21}$$

$$\frac{\partial P}{\partial x} = -\rho\left[\frac{1}{r}\frac{d}{dr}(r\overline{uv})\right] - \left[\frac{1}{r}\frac{d}{dr}(r\tau_{rx})\right] \tag{22}$$

$$\frac{\partial P}{\partial r} = -\rho\left[\frac{1}{r}\frac{d}{dr}(r\overline{v^2})\right] + \rho\frac{\overline{w^2}}{r} - \left[\frac{1}{r}\frac{d}{dr}(r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r}\right] \tag{23}$$

$$\frac{\rho}{r^2}\frac{d}{dr}(r\overline{vw}) + \left[\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta})\right] = 0 \tag{24}$$

where τ_{rx}, τ_{rr}, $\tau_{\theta\theta}$, and $\tau_{r\theta}$ are the nonzero time-averaged components of the stress tensor for power law fluids. Again Krantz and Wasan (12), in the absence of data on non-Newtonian fluids, have used the measurements of Laufer (14) for Newtonian fluids to show that $\tau_{rr} = \tau_{\theta\theta} = \tau_{r\theta} \simeq 0$, and that

$$\tau_{rx} \simeq -K\left|\frac{dU}{dr}\right|^{n-1}\frac{dU}{dr} \tag{25}$$

In prior developments the latter results have been assumed with no theoretical justification.

The boundary conditions on the above equations are

$$\overline{uv} = \frac{dU}{dr} = 0 \text{ at } r = 0$$

$$U = \overline{uv} = \overline{v^2} = \overline{vw} = 0 \text{ at } r = a \tag{26}$$

In view of these boundary conditions Equation (24) can be integrated to obtain $\overline{vw} = 0$ for all r. Differentiating Equation (23) with respect to x yields $\partial P/\partial x = a$ constant. Utilizing these results and Equation (25), one can integrate Equation (22) from the turbulent core to some point in the viscous sublayer to yield the desired relationship between the Reynolds stress and the mean velocity.

$$\overline{uv} = \frac{(y-a)}{2\rho}\frac{\partial P}{\partial x} - \frac{K}{\rho}\left(\frac{dU}{dy}\right)^n \tag{27}$$

An analytical expression is now needed for the mean velocity U in the viscous sublayer. Near the wall the velocities can be expanded in terms of Taylor series:

$$S = \sum_k S_k y^k \tag{28}$$

where

$$S_k = \frac{1}{k!}\left(\frac{\partial^k S}{\partial y^k}\right)_{y=0}; \quad (S = u, v, w, \text{or } U) \tag{29}$$

Therefore the following are obtained for the mean velocity and Reynolds stress distributions in the viscous sublayer (12):

$$U = U_1 y + U_2 y^2 + U_3 y^3 + U_4 y^4 + U_5 y^5 + \ldots \tag{30}$$

$$\overline{uv} = \overline{u_1 v_2} y^3 + \overline{u_2 v_2} y^4 + \ldots . \tag{31}$$

In arriving at the above we have made use of the conditions at the wall and the continuity equation given by Equation (21).

Substituting Equation (30) into Equation (27) and comparing the resulting equation for $\overline{uv}$ with that given by Equation (31), we obtain the following equation for the Reynolds stress in dimensionless form:

$$\overline{uv}^+ = -\left[U_4^+ + \frac{(2n^2 - 3n + 1)}{24}\psi^3\right] 4n(y^+)^3 - \left[5U_5^+ - 4(n-1)U_4^+\psi - \frac{(2^3 - 9n^2 + 10n - 3)}{24}\psi^4\right] n(y^+)^4 \ldots . \tag{32}$$

TABLE 1. RANGE OF PARAMETERS FOR HEAT TRANSFER DATA ON PURELY VISCOUS, NON-NEWTONIAN, TURBULENT PIPE FLOW

Fluids studied*	n-Range	N_{Re} range	N_{Pr} range	δ,† %	σ,† %
Carbopol (4)	0.70 to 0.79	5,145 to 27,130	61.3 to 104	−11.6	12.8
Aqueous CMC (20)	0.66 to 0.92	2,490 to 81,280	34.6 to 434	−0.961	25.9
Corn starch solutions (20)	1.25 to 1.62	3,189 to 11,050	166 to 405	−4.63	13.1
Aqueous CMC (24)	0.56 to 0.94	3,552 to 12,360	104 to 242	−12.9	15.2
Carbopol (24)	0.54 to 0.77	4,404 to 11,690	47.4 to 154	−4.88	12.7
Carbopol (7)	0.45 to 0.87	4,017 to 11,860	10.6 to 149	+2.79	18.2
Attagel clay (7)	0.47 to 0.67	4,570 to 13,660	14.9 to 43.2	−8.76	28.7
Carbopol and corn syrup (7)	0.88	5,035 to 10,570	26.4 to 31.4	−7.29	9.28
Carbopol (26)	0.53 to 0.65	4,258 to 11,550	16.7 to 111	+21.6	27.3
Attagel clay (8)	0.43	4,853 to 6,444	16.9 to 32.9	+14.1	33.7

* Numbers in parentheses refer to literature references.
† Percent arithmetic average deviation and percent standard deviation for predictions of Equation (17).

The corresponding dimensionless equation for the mean velocity is given by

$$U^+ = \left[1 - \frac{\psi}{2}y^+ - \frac{(n-1)}{6}\psi^2(y^+)^2\right] y^+ + U_4^+(y^+)^4 + U_5^+(y^+)^5 + \ldots \quad (33)$$

where

$$\psi \equiv \frac{2(f/2)^{(n-2)/2n}}{[(3n+1)/4](8)^{(n-1)/n}(N_{Re})^{1/n}} \quad (34)$$

The above expressions for the dimensionless Reynolds stress and mean velocity near the wall become identical with those obtained for Newtonian fluids by Wasan, Tien, and Wilke (33) when the flow behavior index is set equal to unity.

The above result demonstrates that the Reynolds stress $\overline{uv}$ for both Newtonian and power law fluids vanishes near the pipe wall with a power of the distance y not less than 3. This agrees with the recent measurements of Sirkar and Hanratty (30) for Newtonian fluids.

The terms in Equations (32) and (33) involving the parameter ψ make a negligible contribution to the value of the local mean velocity; however they can contribute significantly to the value of the local Reynolds stress. The cumulative effect of the terms containing ψ on the Stanton number is relatively minor. It appears that neglecting these terms will result in a maximum error of 5% in the theoretical Stanton number for $N_{Re} \geq 10{,}000$ and 10% for $5{,}000 \leq N_{Re} < 10{,}000$ for the range of flow behavior indices $0.40 \leq n \leq 2.0$.

The coefficients U_4^+ and U_5^+ occurring in Equations (32) and (33) are universal functions of the generalized Reynolds number and the flow behavior index. They can be determined from the value of y^+ at which a smooth and continuous transition to the universal logarithmic velocity profile occurs. This value of y^+ will be denoted by y_c^+ and referred to as the viscous sublayer thickness.

Bogue and Metzner (2) give the following form for the universal logarithmic velocity profile of power law fluids in fully developed turbulent pipe flow:

$$U^+ = 2.42\,ln\,y^+ + I(n, N_{Re}) + c(\xi, f) \quad (35)$$

The function $c(\xi, f)$ has been shown to be insignificant near the wall. The semiempirical function $I(n, N_{Re})$ is given by

$$I(n, N_{Re}) = -2.42\,ln\left\{\left(\frac{3n+1}{8n}\right)[N_{Re}(\sqrt{f/2})^{2-n}8^{n-1}]^{1/n}\right\} \quad (36)$$

This viscous sublayer thickness is found to be given by

$$0.60\,y_c^+ - \frac{3}{20}\psi(y_c^+)^2 - \frac{1}{60}(n-1)\psi^2(y_c^+)^3 = 2.42\,y_c^+ + I(n, N_{Re}) - 1.09 \quad (37)$$

The corresponding universal functions are given by

$$U_4^+ = \frac{3.03}{(y_c^+)^4} - \frac{1}{(y_c^+)^3} + \frac{0.75\psi}{(y_c^+)^2} + \frac{(n-1)\psi^2}{4y_c^+} \quad (38)$$

$$U_5^+ = \frac{3y_c^+ - 9.68 - 2\psi(y_c^+)^2 - 0.5(n-1)\psi^2(y_c^+)^3}{5(y_c^+)^5}$$

The viscous sublayer thickness y_c^+ given by Equation (37) increases as the generalized Reynolds number decreases and the flow behavior index decreases.

Once the viscous sublayer thickness and the universal functions are known, the Reynolds stress and mean velocity distributions can be determined in the wall region from Equations (32) and (33). These in turn can be used to determine the Stanton number given by Equation (17). The definite integral in Equation (17) cannot be solved in closed form. Therefore it was evaluated by numerical integration with a digital computer using Simpson's one-third rule.

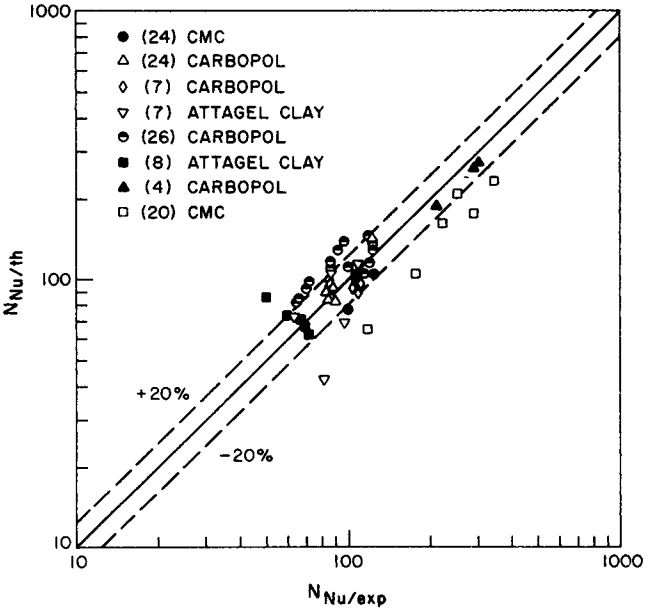

Fig. 1. Theoretical versus experimental Nusselt numbers for heat transfer to highly pseudoplastic fluids in turbulent pipe flow.

DISCUSSION

All the available heat transfer data for the turbulent flow of purely viscous, non-Newtonian fluids, with the exception of those of Farmer (*6*), along with the range of parameters studied, are summarized in Table 1. These include the data of Mizushina and Kuriwaki (*20*), which are the only published data on dilatant fluids ($n > 1$). The data of Friend (*7*), Haines (*8*), and Raniere (*26*) reported here include the corrections made by Petersen (*24*). Insufficient information prevented Petersen from correcting the data of Farmer (*6*), and thus his data are not included here. Whenever possible the isothermal theoretical Nusselt number was corrected for the temperature effect on the fluid viscosity near the wall using the Kreith and Summerfield (*13*) correction $(\mu_w/\mu_b)^{0.10}$ as suggested by Petersen and Christiansen (*25*). Insufficient information prevented making these corrections for the data of Clapp (*4*) and Mizushina and Kuriwaki (*20*).

In order to apply the heat transfer correlations of Metzner and Friend (*19*), Petersen and Christiansen (*25*), and those developed in this paper, it is necessary to know the non-Newtonian friction factor f. In applying these correlations experimental values for the friction factor were used whenever possible. Experimental values were not available for heat transfer data of Clapp (*4*) and those of Mizushina and Kuriwaki (*20*); thus friction factors for these data were determined from the correlation of Dodge and Metzner (*5*).

The percent arithmetic average deviation δ and percent standard deviation σ used in evaluating the heat transfer correlations are defined as

$$\delta \equiv \frac{100}{N} \sum_{i=1}^{N} \left\{ \frac{(N_{Nu})_{\text{theo}} - (N_{Nu})_{\text{exp}}}{(N_{Nu})_{\text{exp}}} \right\}_i \quad (40)$$

$$\sigma \equiv 100 \left[\frac{1}{N-1} \sum_{i=1}^{N} \left\{ \frac{(N_{Nu})_{\text{theo}} - (N_{Nu})_{\text{exp}}}{(N_{Nu})_{\text{exp}}} \right\}_i^2 \right]^{1/2} \quad (41)$$

The data are plotted in Figures 1, 2, and 3 as theoretical Nusselt number given by Equation (17) versus the observed Nusselt number. The Nusselt number is related to the Stanton number in Equation (17) by $N_{Nu} = N_{St} N_{Re}$

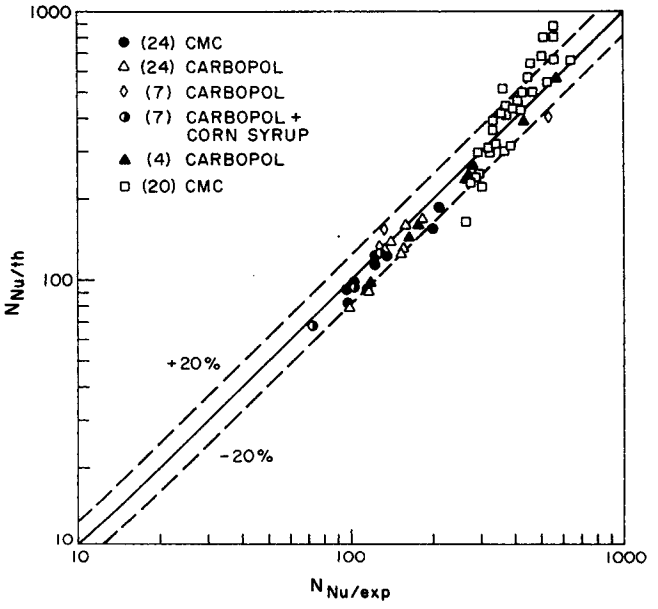

Fig. 2. Theoretical versus experimental Nusselt numbers for heat transfer to slightly pseudoplastic fluids in turbulent pipe flow.

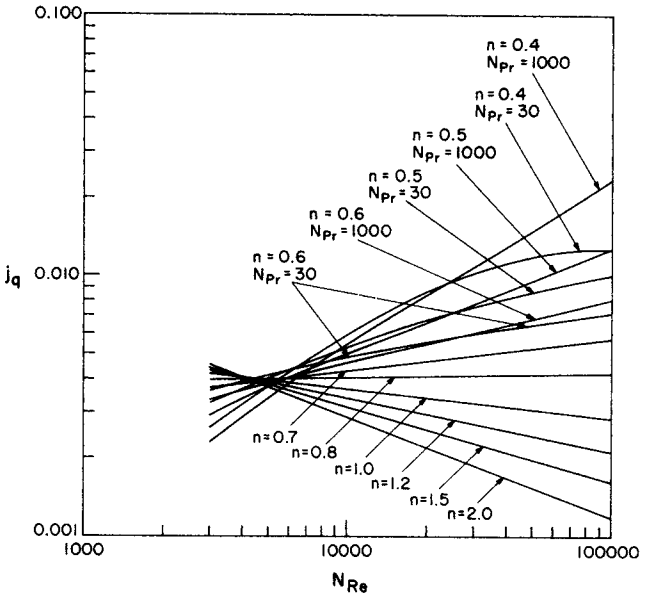

Fig. 3. Theoretical versus experimental Nusselt numbers for heat transfer to dilatant fluids in turbulent pipe flow.

N_{Pr}. The percent arithmetic average deviation δ for the 131 data points in Figures 1, 2, and 3 is -0.77% and the percent standard deviation σ is 21.0%. The theory of Metzner and Friend (*19*) given by Equation (2) predicts the same data with $\delta = +25.0\%$ and $\sigma = 33.7\%$. The modification of Equation (2) by Petersen and Christiansen (*25*) given by Equation (5) yields $\delta = +13.5\%$ and $\sigma = 25.9\%$. The empirical correlation of Clapp (*4*) given by Equation (6) yields $\delta = +26.7\%$ and $\sigma = 36.6\%$; that of Mizushina and Kuriwaki (*20*) given by Equation (7) yields $\delta = +29.8\%$ and $\sigma = 51.3\%$.

The error reported for the correlation developed by the authors is that for the predictions of the exact theory which retains the terms involving ψ in Equations (32) and (33). The effect of this parameter on the Stanton number is small and in most cases can be neglected. In doing so, the percent arithmetic average deviation for the approximate correlation is given by $+3.07\%$ with a percent standard deviation of 20.9%. The approximate correlation offers the advantage of considerably reduced computational time.

The bulk of the heat transfer data on non-Newtonian fluids has been taken on slightly pseudoplastic fluids ($0.70 \leqq n < 1.0$). In order to avoid overemphasizing the data in this range, the data have been broken down into three ranges of flow behavior index. The 57 data points for highly pseudoplastic fluids ($0.43 \leqq n < 0.70$) are shown in Figure 1. The percent arithmetic average deviation for the predictions of Equation (17) for these data is $+2.07\%$, and the percent standard deviation is 24.0%. Slightly pseudoplastic fluid data for 63 sets of parameters are shown in Figure 2, and yield $\delta = -2.67\%$ and $\sigma = 19.3\%$. The relatively few data for dilatant fluids amounting to 11 data points are shown in Figure 3, and yield $\delta = -4.63\%$ and $\sigma = 13.1\%$. The percent arithmetic average deviation and percent standard deviation for the predicted Nusselt number for the various sets of data also are given in Table 1.

One possible explanation for the small increase in the error for the predictions of Equation (17) as the flow behavior index decreases is given here. Equation (37) predicts that the viscous sublayer y_c^+ becomes thicker as the flow behavior index decreases. Only five terms have been retained in the Taylor series expansion of the mean velocity near the wall given by Equation (30). Whereas

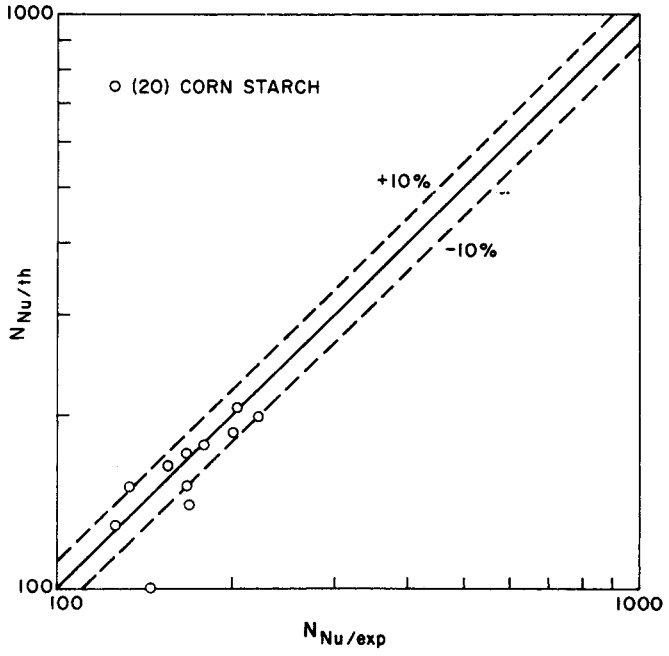

Fig. 4. Theoretical j factor for heat transfer versus generalized non-Newtonian Reynolds number for pseudoplastic and dilatant fluids.

five terms may be sufficient for the thinner viscous sublayers associated with slightly pseudoplastic and dilatant fluids, more terms may have to be retained for highly pseudoplastic fluids. However, the present small increase in error for these fluids does not justify complicating the theoretical development any further.

In applying Equation (17) to the heat transfer data the eddy viscosity for heat was assumed equal to that for momentum (that is, $\beta = 1$). Opfell and Sage (*21*) indicate that β can vary from 1.1 to 1.4 for the turbulent flow of Newtonian fluids in tubes. However, good agreement between theory and heat transfer data for Newtonian fluids has been obtained with $\beta = 1$ (*34*). Reichardt (*27*) suggests that β decreases with decreasing distance from the wall and approaches the value of unity near the wall. The data on Newtonian fluids of Isakoff and Drew (*11*), Sleicher (32), Ludwieg (*16*), and Page et al. (*22, 23*), as well as the physical arguments of Prandtl (*28*), support this conclusion. Since the bulk of the heat transfer resistance is near the wall in the fully developed turbulent flow of high Prandtl number fluids and no data are available for β for power law fluids, it is reasonable to assume a value of unity for β.

A number of the data of Mizushina and Kuriwaki (*20*) and Raniere (*26*) for very pseudoplastic fluids were in the transitional flow regime, rather than in fully developed turbulent flow. However, no correction was made in the theory to account for this. Petersen and Christiansen (25) have shown that transitional flow corrections should be made if

$$2{,}100 \leq (N_{Re})_m < 10{,}000 \tag{43}$$

where

$$(N_{Re})_m \equiv 1.3(3n+1)^2 N_{Re}/[(4n)(n+2^{(n+2)/(n+1)}] \tag{44}$$

They have described a method for correcting for transitional flow which could be applied equally well to the predictions of Equation (17).

Strictly speaking, the analogy theories developed by the present authors and that developed by Metzner and Friend are applicable only to predicting heat transfer coefficients for a constant heat flux at the wall. Nonetheless, the bulk of the heat transfer data was for constant wall temperature. The percent arithmetic average deviation for the predicted Nusselt number for the various sets of data, shown in Table 1, indicates that the constant heat flux assumption is not critical. This conclusion is in agreement with the theory of Siegel and Sparrow (*29*) for heat transfer in the turbulent flow of Newtonian fluids. They were able to conclude on theoretical grounds that the heat transfer mechanism in the thermal entry and fully developed turbulent regions is quite insensitive to these two different conditions at the tube wall for Prandtl numbers greater than approximately 0.7.

The excellent agreement between theory and experiment indicates that the corrections discussed above are relatively minor in most cases. The percent standard deviation of 21.0% for all the data considered is within the experimental error for heat transfer measurement in non-Newtonian fluids.

The new heat transfer correlation for power law fluids given by Equation (17) represent a significant improvement over other heat transfer correlations proposed for these fluids. It has the least bias ($\delta = -0.77\%$) and smallest standard deviation ($\sigma = 21.0\%$) of the various correlations. It also represents the only correlation which does not contain any adjustable parameters which are determined empirically from heat transfer data. The development of the heat and momentum transfer analogy of Metzner and Friend (*19*) is similar to that presented here. However, these authors determine the definite integral in the denominator of Equation (17) from heat transfer data, whereas it is determined analytically in this paper.

The one disadvantage of the new heat transfer correlation presented here for power law fluids is the complexity of Equation (17), although this presents no problem to computer evaluation of the Nusselt number. Sufficiently accurate values of the Nusselt number predicted by Equation (17) may be obtained from the design correlation shown in Figure 4. This is a plot of the j factor for heat transfer defined by

$$j_q \equiv (N_{St})(N_{Pr})^{2/3}\left(\frac{\mu_w}{\mu_b}\right)^{0.10} \tag{45}$$

versus the generalized non-Newtonian Reynolds number for various values of the flow behavior index n. The resulting correlations are nearly independent of the Prandtl number for slightly pseudoplastic, Newtonian, and dilatant fluids. However, for highly pseudoplastic fluids there is a significant dependence on the Prandtl number. Representative curves for Prandtl numbers of 30 and 1,000 are shown for flow behavior indices of 0.4, 0.5, and 0.6 for purposes of interpolation. This marked Prandtl number dependence of highly pseudoplastic fluids indicates that extrapolation of empirical correlations such as Equations (6) and (7) is highly questionable, since these equations predict that the j factor is independent of Prandtl number.

CONCLUSIONS

A new correlation has been developed for heat and mass transfer at large Prandtl numbers in the fully developed turbulent pipe flow of power law fluids. In order to evaluate the resulting expression for the Stanton number, it was necessary to develop analytical expressions for the Reynolds stress and mean velocity distributions in the viscous

sublayer. The equations of turbulent mean motion were used to relate the Reynolds stress to the mean velocity. An analytical expression was developed for the latter by demanding a smooth and continuous transition from the Taylor series expansion of the mean velocity near the wall with the universal mean velocity distribution.

The theoretical predictions were compared with turbulent heat transfer data for both pseudoplastic and dilatant fluids. The data agreed with the new correlation with a percent arithmetic average deviation of -0.77% and a percent standard deviation of 21.0%. The new correlation represents a significant improvement on existing correlations in that it agrees more closely with all the available heat transfer data and has no adjustable parameters determined empirically from heat transfer data.

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NOTATION

a = radius of tube
c = concentration
C_p = heat capacity at constant pressure
f = Fanning friction factor
h = heat transfer coefficient
$I(n, N_{Re})$ = semiempirical function as defined by Equation (36)
j_q = j factor for heat transfer as defined by Equation (43)
k = thermal conductivity
k_c = mass transfer coefficient
K = consistency index as defined by Equation (4)
n = flow behavior index as defined by Equation (4)
N = number of experimental observations
N_{Aw} = mass transfer flux at wall
N_{Nu} = Nusselt number $(2ha/k)$
$(N_{Nu})_{\text{exp}}$ = experimental Nusselt number
N_{Pr} = generalized Prandtl number $\nu\rho C_p/k$
N_{Prc} = Prandtl number as defined by Equation (5)
N_{Prw} = Prandtl number at the wall as defined by Equation (3)
N_{Re} = generalized Reynolds number $(2a)U_a/\nu$
$(N_{Re})_m$ = modified Reynolds number as defined by Equation (44)
N_{St} = Stanton number for heat transfer $(h/\rho C_p U_a)$
N_{Stm} = Stanton number for mass transfer (k_c/U_a)
P = time-averaged local pressure
q = heat flux
q_w = heat flux at the wall
r = radial coordinate
T = temperature
T_a = average temperature
u = fluctuating component of axial velocity
U = time-averaged local axial velocity
U_a = bulk average velocity
U_* = friction velocity $(\tau_w/\rho)^{1/2}$
v = fluctuating component of radial velocity
w = fluctuating component of azimuthal velocity
x = axial coordinate
y = radial coordinate measured from wall
y_c = viscous sublayer thickness

Greek Letters

α = thermal molecular diffusivity
β = ratio of eddy diffusivity for heat to momentum
δ = percent arithmetic average deviation
Δ = rate of strain tensor
ϵ_c = eddy diffusivity for heat
ϵ_v = eddy diffusivity for momentum
θ = azimuthal coordinate
Λ = function as defined by Equation (9)
ν = kinematic viscosity $[(3n+1)/4n]^n (8U_a/2a)^{n-1} K/\rho$
ξ = dimensionless radial coordinate (y/a)
ρ = density
σ = percent standard deviation
τ = viscous stress tensor
τ_{ij} = component of viscous stress tensor representing transfer of j momentum in i direction
τ_w = shear stress at the wall
ϕ = function as defined by Equation (15)
ψ = function as defined by Equation (34)

Superscripts

$\overline{\ }$ = time-averaged quantity
$+$ = dimensionless variable

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A Single-Parameter Equation for Isothermal Vapor-Liquid Equilibrium Correlations

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The Wilson equation is modified so that it contains only one adjustable constant per binary system. In describing binary behavior the new equation is inferior to the original Wilson but superior to the Van Laar equation. In developing ternary behavior from binary data, the new equation seems to be comparable to the original Wilson but definitely superior to the Van Laar equation.

Binary and ternary vapor-liquid equilibrium behavior has been successfully predicted from one γ° value per binary only.

Several expressions for the dependence of the activity coefficient on composition for binary and multicomponent systems have been suggested in the literature (*1, 2 10, 12, 17, 21, 25, 27*). Orye and Prausnitz (*17*) have shown that the Wilson equation (*27*) is superior to the Van Laar equation (*17, 20*), both in describing binary vapor-liquid equilibrium and in predicting multicomponent behavior from binary data only. More recently, Heil and Prausnitz (*12*) and Renon and Prausnitz (*21*) proposed two new expressions, based essentially on Wilson's concept of "local volume fractions," which seem to be superior to the original Wilson equation in describing highly nonsymmetrical and partially miscible systems (the Wilson equation cannot describe immiscible systems). All these equations contain a minimum of two adjustable constants per binary system, and therefore at least a pair of activity coefficients is needed for their evaluation so that the whole concentration range can be described. Furthermore, the constants are essentially of empirical nature and hence they cannot be predicted from pure component properties only.

A modified form of the Wilson equation, containing only one adjustable constant per binary system, is presented in this study. The new equation has been successfully tested both in describing binary vapor-liquid equilibrium and in predicting ternary behavior from binary data only. The new expression has also been compared with the original Wilson and Van Laar equations. In describing binary equilibrium, it is inferior to the Wilson, as should be expected, but better than the Van Laar equation. In predicting ternary equilibrium from binary data only, it appears to be as good as the Wilson equation and definitely superior to that of Van Laar. It can also describe the vapor-liquid equilibrium behavior of a binary system, based on the value of the activity coefficient of the one